



Qualifying Examination



Academic Year 2026 (2569) : PART 1 : ROUND 1

7 August 2026

Name:.....

Student ID:.....

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 8 questions .
2. Write down detailed and clear solutions to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	
7	
8	

1 Vector Calculus

Question 1.

Consider three-dimensional flat space with Cartesian coordinates (x^1, x^2, x^3) . Let $\vec{r} = x^1\hat{e}_1 + x^2\hat{e}_2 + x^3\hat{e}_3$, and $r \equiv \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$. In this problem, $m \geq 0$.

1.1) [1 point.] Compute

$$\frac{d}{dr} \left(\frac{e^{-mr}}{r^n} \right),$$

where $n \geq -1$.

1.2) [3 points.] Show that

$$(\nabla^2 - m^2) \left(\frac{e^{-mr}}{r} \right) = 0,$$

for $r > 0$. Hint: you may find it useful to denote $T_n = e^{-mr}/r^n$.

1.3) [2 points.] Evaluate

$$\int_{\mathcal{V}_\epsilon} dV \, m^2 \frac{e^{-mr}}{r}, \quad \epsilon > 0,$$

where dV is a volume element of $\mathcal{V}_\epsilon = \{(x^1, x^2, x^3) \mid 0 \leq \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2} \leq \epsilon\}$.

1.4) [2 points.] With the help of divergence theorem, evaluate

$$\int_{\mathcal{V}_\epsilon} dV \, \nabla^2 \left(\frac{e^{-mr}}{r} \right).$$

1.5) [1 point.] By using an ansatz

$$(\nabla^2 - m^2) \left(\frac{e^{-mr}}{r} \right) = \lambda \delta^{(3)}(\vec{r}),$$

determine λ .

1.6) [1 point.] Argue whether the final result is what you expected from physics. Hint: consider an appropriate limit of the parameter m .

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2 Linear Algebra

Question 2.

A matrix N is called nilpotent if $N^m = 0$ for some positive integer m .

2.1) [2 points.] Show that all eigenvalues of a nilpotent matrix are zero.

2.2) [3 points.] Show that if N is a nilpotent, $n \times n$ matrix, then $N^n = 0$.

2.3) [5 points.] Let $A = \lambda I + N$, where $0 \neq \lambda \in \mathbb{C}$ and N is nilpotent. Show that A^{-1} can be written as a polynomial in A of degree at most $n - 1$.

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3 Group Theory/Lie Algebra

Question 3.

The Heisenberg group $H_3(\mathbb{R})$ is a continuous matrix group that plays a foundational role in quantum mechanics. Consider the set H of all 3×3 matrices with real entries of the form

$$M(x, y, z) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix},$$

for $x, y, z \in \mathbb{R}$.

3.1) [3 points.] Verify that the set H satisfies the **closure axiom** under ordinary matrix multiplication and that every element of H has an **inverse** in H .

3.2) [1 point.] Is the group H abelian? Justify your answer.

3.3) [3 points.] The center of a group, denoted $Z(H)$, consists of all elements in the group that commute with every element in the group ($gc = cg$ for all $g \in H, c \in Z(H)$). Determine the general form of the elements in the center of the Heisenberg group.

3.4) [3 points.] Identify the general matrix form of an element C in the Heisenberg **Lie algebra** $\mathfrak{h}$, and show that the commutator $[A, B]$ of any two elements $A, B \in \mathfrak{h}$ belongs to $\mathfrak{h}$.

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4 Group Theory/Lie Algebra

Question 4

In a $1 + 2$ dimensional Minkowski spacetime, the metric is given by $g = \text{diag}(-1, 1, 1)$, corresponding to coordinates (t, x, y) . The continuous symmetries of this spacetime are described by the Lorentz group and its associated Lie algebra, $so(1, 2)$.

4.1) [3 points.] A vector field ξ generates a flow of isometries if it satisfies the Killing equation

$$\xi^k \partial_k g_{ij} + \partial_i \xi^k g_{kj} + \partial_j \xi^k g_{ik} = 0.$$

Consider the vector field $\xi = x\partial_t + t\partial_x$. Explicitly evaluate the Killing equation for all relevant non-zero components to prove that ξ is a Killing vector field.

4.2) [4 points.] The Lie algebra $so(1, 2)$ consists of matrices ω that satisfy $(g\omega)^T = -g\omega$. One of the basis matrices for this algebra is given by:

$$\omega_3 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

A Lorentz transformation Λ can be obtained by exponentiating the Lie algebra element: $\Lambda(\lambda) = e^{\lambda\omega_3}$. Derive the explicit 3×3 matrix for $\Lambda(\lambda)$ in closed form.

4.3) [3 points.] Let A and B be two matrices belonging to $so(1, 2)$, meaning they satisfy $A^T g = -gA$ and $B^T g = -gB$. Show directly that $(g[A, B])^T = -g[A, B]$. That is, the commutator $[A, B]$ also belongs to the Lie algebra.

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5 Function of Complex Variables

Question 5.

5.1) [4 points.] Let D be an open subset of $\mathbb{C}$ and $f : D \rightarrow \mathbb{C}, u : D \rightarrow \mathbb{R}, v : D \rightarrow \mathbb{R}$. The function f is analytic in D if it can be expressed as $f(x, y) = u(x, y) + iv(x, y)$, where u and v have continuous partial derivatives $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y$ which satisfy Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Consider the case where $D = \mathbb{C} \setminus \{-1\}$, and

$$u(x, y) = \frac{x+1}{(x+1)^2 + y^2}.$$

Find $v(x, y)$ which makes $f(x, y)$ analytic in D , and that $v(0, 0) = i$.

5.2) [6 points.] Rigorously evaluate the integral

$$I(k) = \int_{-\infty}^{\infty} \exp(-ax^2 + ikx) dx \quad (a, k > 0)$$

using complex contour integration along a suitable contour. You may use the standard real result

$$\int_{-\infty}^{\infty} e^{-au^2} du = \sqrt{\frac{\pi}{a}}$$

without proof. Hint: after changing the variable, carefully check whether the integration path is along the real line.

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6 Integral Transforms

Question 6. The Laplace transform of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt.$$

6.1) [3 points] Find Laplace transform of the following functions:

1. t^n
2. $\cos(t)$
3. $\sin(t)$

6.2) [2 points] Show that the Laplace transform of the first derivative, $f'(t)$ is given by

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

where $f'(t) = \frac{df(t)}{dt}$.

6.3) [5 points] By using the Laplace transform method, Find the solution $x(t)$ for the ODE

$$x'(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t)$$

in terms of $x(0)$.

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7 Integral Transforms

Question 7. The Fourier transform of a function $f(\mathbf{r})$ is defined by

$$\tilde{f}(\mathbf{k}) = \mathcal{F}\{f(\mathbf{r})\} = \int_{\mathbb{R}^3} e^{-i\mathbf{k}\cdot\mathbf{r}} f(\mathbf{r}) d^3r,$$

with inverse transform

$$f(\mathbf{r}) = \frac{1}{(2\pi)^3} \int_{\mathbb{R}^3} e^{i\mathbf{k}\cdot\mathbf{r}} \tilde{f}(\mathbf{k}) d^3k.$$

7.1) [2 points] Show that the Fourier transform of the Laplacian is given by

$$\mathcal{F}\{\nabla^2 f(\mathbf{r})\} = -k^2 \tilde{f}(\mathbf{k}).$$

7.2) [8 points] By using the Fourier transform, show that the solution of Poisson's equation

$$\nabla^2 \phi(\mathbf{r}) = -\frac{\rho(\mathbf{r})}{\varepsilon_0},$$

for the point charge density

$$\rho(\mathbf{r}) = q \delta^{(3)}(\mathbf{r}),$$

is

$$\phi(\mathbf{r}) = \frac{q}{4\pi\varepsilon_0 r}.$$

You may use the identity

$$\int_0^\infty \frac{\sin x}{x} dx = \frac{\pi}{2}.$$

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8 Ordinary Differential Equations

Question 8. The time-independent Schrödinger equation for a particle confined in a two-dimensional infinite square well is

$$-\frac{\hbar^2}{2m} \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y^2} \right) = E\psi,$$

where

$$0 < x < L, \quad 0 < y < L,$$

7.1) [3 points] Use a method of separation to show that the Schrödinger equation can be separated into the following ordinary differential equations:

$$X''(x) + \lambda^2 X(x) = 0, \quad \text{and} \quad Y''(y) + \mu^2 Y(y) = 0$$

How are the constants λ and μ related?

7.2) [7 points] Solve the differential equations obtained in part 7.1 subject to the boundary conditions. Then show that the normalized wavefunction is

$$\psi_{nm}(x, y) = \frac{2}{L} \sin\left(\frac{n\pi x}{L}\right) \sin\left(\frac{m\pi y}{L}\right),$$

and that the corresponding energy eigenvalues are

$$E_{nm} = \frac{\pi^2 \hbar^2}{2mL^2} (n^2 + m^2), \quad n, m = 1, 2, 3, \dots$$

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