



Qualifying Examination



Academic Year 2023 (2566) : PART 1 : ROUND 2

24 JAN 2024

Name:.....

Student ID:.....

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

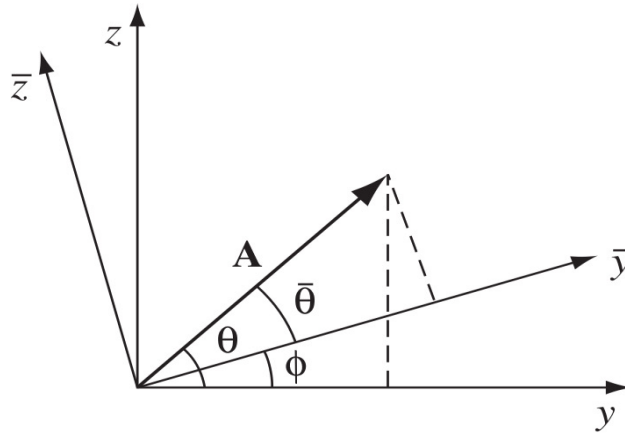
1. Choose 4 out of the following 8 questions .
2. Write down detailed and clear solution to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	
7	
8	

1 Vector Calculus

Question 1.

From the figure



$$A_y = A \cos \theta, \text{ and } A_z = A \sin \theta$$

- 1.1) Write down the components of $\bar{\mathbf{A}}$ in the rotation axes and in the matrix form.
- 1.2) Show that the two-dimensional rotation matrix preserves dot product. (Hint: show that $\bar{A}_y \bar{B}_y + \bar{A}_z \bar{B}_z = A_y B_y + A_z B_z$)

[10 marks]

Name:.....

Name:.....

2 Linear Algebra

Question 2.

1. Solve the following set of linear equations using a matrix approach:

$$2x - 5y + 3z = 7$$

$$3x - 2y + 4z = 5$$

$$x + 3y + 2z = -2$$

Solving them using the Gaussian elimination, invert matrix or Cramer's rule are all acceptable. [5 points]

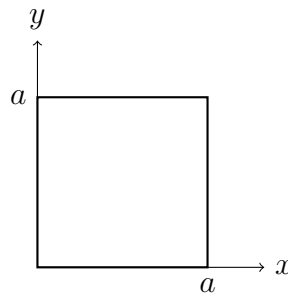


Figure 1: A thin square plate

2. In rigid body problems, it is useful to calculate a tensor of inertia I . For a real symmetric tensor I , there always exists an orthogonal matrix R which diagonalises the tensor I giving

$$I_D = \begin{pmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{pmatrix} = RIR^T.$$

Now let's consider a thin square plate whose tensor of inertia is

$$I = \frac{ma^2}{12} \begin{pmatrix} 4 & -3 & 0 \\ -3 & 4 & 0 \\ 0 & 0 & 8 \end{pmatrix}.$$

Diagonalise the provided matrix to find out the eigenvalues (I_1, I_2, I_3) as well as the transformation matrix R . [5 points]

Name:.....

Name:.....

3 Group Theory/Lie Algebra

Question 3.

The Lorentz group is defined as the set of transformations $x^\mu \mapsto \lambda^\mu{}_\nu x^\nu$, which leaves invariant the quantity

$$\eta_{\mu\nu} x^\mu x^\nu = t^2 - x^2 - y^2 - z^2.$$

3.1) Show that, equivalently, Λ such that $\eta = \Lambda^T \eta \Lambda$ is a Lorentz transformation. [1 point]

3.2) As is well known, the infinitesimal generators of the Lorentz group divide into two types, those that generate rotations and those that generate boosts:

$$J^j = i\epsilon_{jkl} x_k \partial_l, \quad K^j = -i(x_0 \partial_j + x_j \partial_0), \quad (1)$$

where 0 is the time component and $j, k = 1, 2, 3$.

How are the K^j 's generators of Lorentz boost? Let

$$B_x(\eta) = \exp(-i\eta K^1) = \exp[-\eta(t\partial_x + x\partial_t)].$$

Show that its action for an arbitrary function $\psi(t, x)$ of the spacetime coordinates (reducing to two dimension for simplicity) is

$$B(\eta)\psi(t, x) = \psi(t \cosh \eta - x \sinh \eta, x \cosh \eta - t \sinh \eta).$$

Hint: Start from the expression $K^1 = -i\frac{\partial}{\partial \xi}$, where $t = r \sinh \xi$ and $x = r \cosh \xi$. (You don't have to derive this expression.)

Some useful identities:

$$\begin{aligned} \sinh(\xi - \eta) &= \sinh \xi \cosh \eta - \cosh \xi \sinh \eta, \\ \cosh(\xi - \eta) &= \cosh \xi \cosh \eta - \sinh \xi \sinh \eta. \end{aligned}$$

[3 points]

3.2) Compute the commutators between K^j and all the six generators (1). Do the set of all boosts form a group? [6 points]

Name:.....

Name:.....

4 Group Theory/Lie Algebra

Question 4.

Let $\mathfrak{g}$ be a matrix Lie algebra, and $[X, Y] \equiv XY - YX$ be the matrix commutator between elements X, Y in the Lie algebra.

4.1) Show that the map ad defined by $\text{ad}_X(Y) \equiv [X, Y]$ is a representation of the Lie algebra. This is called the *adjoint representation*. [2 points]

4.2) Now specialize to the Lie algebra $\mathfrak{su}(2)$ generated by the three Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

which obey the commutation relation (assuming Einstein summation convention)

$$[\sigma_j, \sigma_k] = 2i\epsilon_{jkl}\sigma_l.$$

Let $|j\rangle = \sigma_j$, $j = 1, 2, 3$ be the orthonormal basis of the adjoint representation. Write down the matrix representation of ad_{σ_1} , ad_{σ_2} and ad_{σ_3} in this basis. Are they Hermitian? Are they unitary? [3 points]

4.3) Compute $\exp(i\lambda \text{ad}_{\sigma_j}/2)$ for $j = 1, 2, 3$ by directly exponentiating the matrix. Verify that $\exp(i\lambda \text{ad}_{\sigma_3}/2)|1\rangle$ gives the same result as $e^X \sigma_1 e^{-X}$, where $X = i\lambda \sigma_3/2$. (This is the special case of the identity $e^X Y e^{-X} = e^{\text{ad}_X}(Y)$.)

[5 points]

Name:.....

Name:.....

5 Function of Complex Variables

Question 5.

1. Find all the roots of $z^4 = 1$. [2 points]
2. Let $f : \mathbb{C} \rightarrow \mathbb{C}$, $u : \mathbb{C} \rightarrow \mathbb{R}$, $v : \mathbb{C} \rightarrow \mathbb{R}$. The function f is analytic if it can be expressed as $f(x, y) = u(x, y) + iv(x, y)$, where u and v satisfies Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Consider the case where $u(x, y) = x^3 - 3xy^2$. Find $v(x, y)$ which makes $f(x, y)$ analytic. [3 points]

3. With the help of contour integral and residue theorem, evaluate

$$\int_{-\infty}^{\infty} \frac{dx}{x^2 + 1}.$$

[5 points]

Name:.....

Name:.....

6 Integral Transforms

Question 6.

6.1). Let us consider the following function:

$$\psi(x) = \frac{1}{\sqrt{2a}} [\Theta(x+a) - \Theta(x-a)],$$

where $a > 0$ and $\Theta(x)$ is the Heaviside step function, namely, $\Theta(x < 0) = 0$ and $\Theta(x > 0) = 1$. Find the Fourier transform of $\psi(x)$, namely,

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \psi(x) e^{-ikx}.$$

[7 marks.]

6.2). Is the function $\psi(x)$ normalized to unity? In other words, is $\psi(x)$ a square-integrable function with norm equal to one?

[1 mark.]

6.3). But furthermore, is also $\phi(k)$ a square-integrable function with norm equal to one? Please justify your answer. Hint: You may find some of the following integrals useful:

$$\int_0^{+\infty} du \frac{\sin(u)}{u} = \frac{\pi}{2}, \quad \int_0^{+\infty} du \frac{\sin(u)}{u^2} = \infty, \quad \int_0^{+\infty} du \frac{\sin^2(u)}{u} = \infty, \quad \int_0^{+\infty} du \frac{\sin^2(u)}{u^2} = \frac{\pi}{2}.$$

[2 marks.]

Name:.....

Name:.....

7 Integral Transforms

Question 7.

In synchronous gauge the line element is

$$ds^2 = a^2(\tau) \left[-d\tau^2 + (\delta_{ij} + h_{ij}) dx^i dx^j \right],$$

where τ is the conformal time and $a(\tau)$ is the scale factor. The metric perturbation h_{ij} can be decomposed into a trace part $h \equiv h_{ii}$ and a traceless part $h_{ij}^{\parallel}$ where $h_{ij} = h\delta_{ij}/3 + h_{ij}^{\parallel}$. $h_{ij}^{\parallel}$ can be written in terms of a scalar field μ ,

$$h_{ij}^{\parallel} = \left(\partial_i \partial_j - \frac{1}{3} \delta_{ij} \nabla^2 \right) \mu.$$

We will be working in Fourier space k and introduce two scalar fields $h(\tau, \mathbf{k})$ and $\eta(\tau, \mathbf{k})$. We can write the scalar mode of h_{ij} as a Fourier integral

$$h_{ij}(\tau, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\mathbf{k}\cdot\mathbf{x}} \left[\hat{\mathbf{k}}_i \hat{\mathbf{k}}_j h(\tau, \mathbf{k}) + \left(\hat{\mathbf{k}}_i \hat{\mathbf{k}}_j - \frac{1}{3} \delta_{ij} \right) 6\eta(\tau, \mathbf{k}) \right],$$

$$\mathbf{k} = k \hat{\mathbf{k}}.$$

Note that h denotes the trace of h_{ij} in both the real space and Fourier space.

7.1) Write $\mu(\tau, \mathbf{k})$ in terms of $h(\tau, \mathbf{k})$ and $\eta(\tau, \mathbf{k})$. [5 points]

7.2) The component-00 of the perturbed Einstein equation is

$$\frac{1}{2} \partial^i \partial^j h_{ij}^{\parallel} - \frac{1}{3} \nabla^2 h + \mathcal{H} h' = 8\pi G a^2 \delta\rho,$$

where $\mathcal{H} \equiv \frac{1}{a} \frac{da}{d\tau}$ and $'$ denotes derivative with respect to τ . Show that in Fourier space it is equivalent to

$$k^2 \eta - \frac{1}{2} \mathcal{H} h' = -4\pi G a^2 \delta\rho. \quad [5 \text{ points}]$$

Name:.....

Name:.....

8 Ordinary Differential Equations

Question 8.

8.1). Find a real general solution to the following ordinary differential equation:

$$\frac{d^4}{dx^4}y(x) - \lambda^4 y(x) = 0,$$

where λ is a positive constant.

[5 marks.]

8.2). Impose the following boundary conditions on the general solution you wrote in the previous question:

$$y(x) = 0, \quad \frac{d^2}{dx^2}y(x) = 0, \quad \text{at } x = 0 \quad \text{and} \quad x = L > 0.$$

What solutions did you find?

[5 marks.]

Name:.....

Name:.....
