



Qualifying Examination



Academic Year 2025 (2568) : PART 1 : ROUND 2

19 JAN 2026

Name:.....

Student ID:.....

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 8 questions .
2. Write down detailed and clear solution to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	
7	
8	

Name:.....

1 Vector Calculus

Question 1.

Prove the following identities:

1.1) [4 points.]

$$\nabla \cdot (\mathbf{a} \times \mathbf{b}) = \mathbf{b} \cdot (\nabla \times \mathbf{a}) - \mathbf{a} \cdot (\nabla \times \mathbf{b}).$$

1.2) [6 points.]

$$\mathbf{a} \times (\nabla \times \mathbf{a}) = \frac{1}{2} \nabla |\mathbf{a}|^2 - (\mathbf{a} \cdot \nabla) \mathbf{a}.$$

Name:.....

Name:.....

2 Linear Algebra

Question 2.

Consider the following linear transformations on $\mathbb{R}^3$:

$$T(x, y, z) = (x + 2y + 3z, 2y + 3z, 3z), \quad S(x, y, z) = (x + y, y + z, z).$$

2.1) [2 points.] Write the matrix representation of T and S in the standard basis.

2.2) [5 points.] Are the matrices diagonalizable? Justify your answer. Note that the eigenvectors do not necessarily have to be orthogonal for a matrix to be diagonalizable.

2.3) [3 points.] Is S invertible? If so, find S^{-1} .

Name:.....

Name:.....

3 Group Theory/Lie Algebra

Question 3.

Let D_8 be the symmetry group of a square. The group is generated by a rotation r and a reflection s satisfying

$$r^4 = e, \quad s^2 = e, \quad srs = r^{-1},$$

where e denotes the identity element. The group has eight elements.

3.1) [5 points.] Recall that two elements x, y of a group G are said to be *conjugate* or belong in the same *conjugacy class* if there exists $g \in G$ such that $y = gxg^{-1}$. **Show** that the four reflections in D_8 separate into two distinct conjugacy classes.

3.2) [5 points.] It is possible to embed D_8 into a larger group in such a way that all reflections are conjugate. Consider the action of a group G on itself by left multiplication:

$$\begin{aligned} G \times G &\rightarrow G \\ (g, h) &\mapsto gh. \end{aligned}$$

This action associates to each element $g \in G$ a permutation of the set G . In this way, D_8 can be regarded as a subgroup of the permutation group of eight objects, S_8 . **Show** that all reflections in D_8 become conjugate when viewed as elements of S_8 . You do **not** need to explicitly find permutations in S_8 that conjugate these reflections.

Hint: Write permutations as products of disjoint cycles. You can use the fact that two permutations are conjugate if and only if they have the same decomposition into cycles (for example, if both are products of four disjoint transpositions).

Name:.....

Name:.....

4 Group Theory/Lie Algebra

Question 4.

Consider the unitary group

$$U(N) = \{U \in \mathbb{C}^{N \times N} | U^\dagger U = I\},$$

and let us denote the (real) Hilbert-Schmidt inner product between two complex matrices by

$$\langle X, Y \rangle = \operatorname{Re} \operatorname{Tr}(X^\dagger Y).$$

4.1) [3 points.] The Lie algebra of a Lie group can be identified with the set of tangent vectors to smooth curves passing through the identity element. Let $U(t) \in U(N)$ be a smooth curve with $U(0) = I$, and define the tangent vector

$$X \equiv \left. \frac{dU}{dt} \right|_{t=0}.$$

Show that the Lie algebra $\mathfrak{u}(N)$ of $U(N)$ consists of all $N \times N$ anti-Hermitian matrices. That is, $X^\dagger = -X$.

4.2) [3 points.] Fix $U \in U(N)$. For any $X \in \mathfrak{u}(N)$, define a smooth curve $U_X(t) \equiv e^{tX}U$ on the unitary group. Consider the function

$$f(U) = \operatorname{Re} \operatorname{Tr}(A^\dagger U B U^\dagger),$$

where $A, B \in \mathbb{C}^{N \times N}$ are fixed matrices. Such expression frequently arises in quantum mechanics as expectation values or cost functions that depend on a unitary transformation. **Compute** the directional derivative of f at U :

$$Df(U)[X] \equiv \left. \frac{d}{dt} f(U_X) \right|_{t=0}.$$

4.3) [4 points.] In order to minimize (or maximize) the function $f(U)$, it is natural to move in the direction of steepest descent (or ascent). Analogous to the Euclidean case, the *Riemannian gradient* at U is defined to be the unique tangent vector $\nabla f(U) \in \mathfrak{u}(N)$ such that

$$Df(U)[X] = \langle \nabla f(U), X \rangle,$$

Name:.....

for all $X \in \mathfrak{u}(N)$. **Compute** $\nabla f(U)$ explicitly for the function defined in **4.2**).

Name:.....

Name:.....

5 Function of Complex Variables

Question 5.

Consider the integral

$$\int_0^\infty \frac{\cos mx}{4x^4 + 5x^2 + 1} dx,$$

for $m > 0$.

5.1) [2 points.] By applying analytic continuation, show that the solution to the above integral could be found by evaluating the contour integral;

$$\oint f(z) dz,$$

where the contour is taken over the upper half-plane. $z = x + iy$ and

$$f(z) = \frac{e^{imz}}{(z + i)(z - i)(2z + i)(2z - i)}.$$

5.2) [2 points.] Identify the poles of the above contour integral.

5.3) [2 points.] Show that all the conditions for Jordan's lemma are satisfied.

5.4) [4 points.] By using Cauchy's residue theorem, show that

$$\int_0^\infty \frac{\cos mx}{4x^4 + 5x^2 + 1} dx = \frac{\pi}{6} (4e^{-m/2} - e^{-m}).$$

Name:.....

Name:.....

6 Integral Transforms

Question 6.

A Fourier transform of a function $f(x)$ is given by

$$\mathcal{F}(f(x)) = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{ikx}$$

6.1) [3 points.] show that the Fourier tranform of the n -derivative of $f(x)$ is

$$\mathcal{F}\left(\frac{d^n f(x)}{dx^n}\right) = (-ik)^n F(k).$$

6.2) [5 points.] By taking the Fourier transform of the Schrodinger's equation

$$-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \psi(x, t) = i\hbar \frac{\partial}{\partial t} \psi(x, t),$$

show that its solution, $\psi(x, t)$ can be written as

$$\psi(x, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dk \tilde{\psi}(k, 0) \exp\left(-i\left(kx + \frac{\hbar k^2}{2m}t\right)\right)$$

where $\tilde{\psi}(k, t)$ is the Fourier transform of $\psi(x, t)$.

6.3) [2 points.] If at the initial time $t = 0$, the wavefunction $\psi(x, 0) = \delta(x)$. Calculate the form of $\psi(x, t)$. (You may leave the solution unintegrated)

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Name:.....

7 Integral Transforms

Question 7.

A series circuit contains a resistor R , a capacitor C , and a battery with a time-varying voltage $V(t)$. The charge $q(t)$ on the capacitor satisfies the differential equation

$$R\frac{dq}{dt} + \frac{q}{C} = V(t),$$

where the capacitor is initially uncharged, $q(0) = 0$. Assume the applied voltage is

$$V(t) = V_0 \sin(\omega t).$$

7.1) [3 points.] The Laplace transform of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

Using this definition, find the Laplace transforms of the following functions:

$$\cos(\omega t), \quad \sin(\omega t), \quad e^{-at},$$

where $\omega > 0$ and $a > 0$.

7.2) [2 points.] Let $F(s) = \mathcal{L}\{f(t)\}$. Show that the Laplace transform of the derivative $f'(t)$ is given by

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

7.3) [5 points.] Apply the Laplace transform to the differential equation

$$R\frac{dq}{dt} + \frac{q}{C} = V_0 \sin(\omega t),$$

and solve for the charge $q(t)$ on the capacitor. (You may use the results from part (a) for the inverse Laplace transforms.)

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Name:.....

8 Ordinary Differential Equations

Question 8.

Consider a general homogeneous second-order ordinary differential equation

$$y'' + P(x)y' + Q(x)y = 0,$$

where $y = y(x)$. One method for solving such equations is the *series method*, in which the solution is assumed to take the form

$$y(x) = \sum_{j=0}^{\infty} a_j (x - x_0)^{s+j}, \quad \text{where } a_0 \neq 0,$$

where x_0 is the point about which the solution is expanded. The point $x = x_0$ must be a regular point, meaning that both $P(x)$ and $Q(x)$ are finite at $x = x_0$.

For computational convenience, solutions are often expanded about $x_0 = 0$. However, if $x = 0$ is a singular point of the differential equation, it is useful to investigate the solution near infinity. This can be done by introducing the change of variable

$$u = \frac{1}{x},$$

and seeking a series solution of the form

$$y(u) = \sum_{j=0}^{\infty} a_j u^{s+j}, \quad a_0 \neq 0.$$

Now consider the following linear second-order ordinary differential equation:

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0.$$

8.1) [2 points.] Rewrite the differential equation in terms of the variable $u = 1/x$.

8.1) [8 points.] Apply a series method in the variable u to find a solution of the differential equation.

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