



Qualifying Examination



Academic Year 2025 (2568) : PART 2 : ROUND 2

21 JAN 2026

Name:.....

Student ID:.....

Part 2: Subjects Relating to student's Research Field : General Relativity and Black Hole Physics

Instructions and cautions

1. Choose 3 out of the following 6 questions relating student's field.
2. Write down detailed and clear solution to problems of your choice.
3. Complete answer are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	

1 Manifold, Curvature, Gravitation, Einstein equation

Question 1.

1.1) [3 points.] Writing down the symmetric, anti-symmetric, and cyclic properties for the Riemann curvature tensor. (Do not need to prove.)

1.2) [2 points.] By definition, $[\nabla_\mu, \nabla_\nu] A^\alpha = R^\alpha_{\beta\mu\nu} A^\beta$. Show that

$$[\nabla_\mu, \nabla_\nu] A_\alpha = -R^\beta_{\alpha\mu\nu} A_\beta.$$

and write down

$$[\nabla_\mu, \nabla_\nu] T_{\alpha\beta} = ?$$

Where $T_{\alpha\beta}$ is a general rank-2 tensor.

1.3) [5 points.] Write down and prove the (differential) Bianchi identity by using the results in **1.1** and **1.2**.

(Hint: You may start with considering $T_{\alpha\beta}$ in **1.2** as $\nabla_\alpha V_\beta$, taking the cyclic combination as $[\nabla_\mu, \nabla_\nu] \nabla_\alpha V_\beta + [\nabla_\nu, \nabla_\alpha] \nabla_\mu V_\beta + [\nabla_\alpha, \nabla_\mu] \nabla_\nu V_\beta$ and rearranging in the form of $\nabla_\mu [\nabla_\nu, \nabla_\alpha] V_\beta + \dots$)

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2 Field theory of gravity

Question 2. Consider a scalar field in a spherically symmetric spacetime. The field equation and the background spacetime are given by

$$\nabla_\mu \nabla^\mu \Psi(t, r, \theta, \phi) = [\partial_\mu \partial^\mu + \Gamma^\mu_{\mu\rho} \partial^\rho] \Psi(t, r, \theta, \phi) = 0.$$

$$ds^2 = -f(r) dt^2 + f(r)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2.$$

2.1) [4 points.] Derive the equation of motion for the scalar field in this spacetime.

2.2) [4 points.] Take $\Psi(t, r, \theta, \phi) = e^{-i\omega t} \bar{\Psi}(r, \theta, \phi)$, separate the angular part by using the eigen-equation for the spherical harmonics, and write down the radial equation. The eigen-equation for the spherical harmonics is given by

$$[\partial_\theta^2 + \cot \theta \partial_\theta + \sin^{-2} \theta \partial_\phi^2] Y^{\ell m}(\theta, \phi) = -\ell(\ell + 1) Y^{\ell m}(\theta, \phi).$$

2.3) [2 points.] Simplify the radial equation into the Schrödinger-like form by defining the tortoise coordinate as $\partial_{r_\star} = f(r) \partial_r$.

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3 Hamiltonian formalism for General Relativity

Question 3.

Let us consider the Einstein-Hilbert action in 3+1 formalism:

$$S = \frac{1}{2} \int dt dx^3 N \sqrt{h} (K_{ij} K^{ij} - K^2 + \mathcal{R}) , \quad (1)$$

where we set $8\pi G = 1$, N is a lapse function, K_{ij} is the extrinsic curvature of spacelike hypersurfaces, $K \equiv h^{ij} K_{ij}$, h_{ij} is the induced metric, and $\mathcal{R}$ is Ricci scalar on the hypersurfaces.

3.1) [1 points.] Compute the momentum Π^{ij} conjugated to h_{ij} in terms of K_{ij} and K

3.2) [2 points.] Write K_{ij} in terms of Π_{ij} and its trace

3.3) [4 points.] Compute the Hamiltonian density of the action in Eq. (1)

3.4) [3 points.] Compute the Hamiltonian and momentum constraint equations for General Relativity

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4 Quantum Cosmology

Question 4.

4.1) [5 points.] Compute the Hamiltonian and momentum constraint equations for the case where a scalar field appears in the gravitational field described by General Relativity

4.2) [5 points.] Use the canonical quantization to find the wave equations for the gravitational field and scalar field

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5 Black hole solutions

Question 5.

For the static spacetime with spherical symmetry, the general form of the metric can be characterized by 2 functions as

$$ds^2 = -e^{2\alpha(r)} dt^2 + e^{2\beta(r)} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2),$$

and the non-zero independent components of the Ricci tensor can be written as

$$R_{00} = -e^{2(\alpha-\beta)} \left(\alpha'^2 - \beta' \alpha' + \alpha'' + \frac{2\alpha'}{r} \right), \quad (2)$$

$$R_{11} = \alpha'^2 - \beta' \alpha' + \alpha'' - \frac{2\beta'}{r}, \quad (3)$$

$$R_{22} = -e^{-2\beta} (-r\alpha' + e^{2\beta} + r\beta' - 1). \quad (4)$$

If we ignore the static condition, explain in detail as follows

5.1) [5 points.] What is the number of independent functions [1] and show how to obtain such the metric as well as explain how it differs from one with static condition [4].

5.2) [5 points.] Explain how the components of the Ricci tensor differ from those with the static condition [2] and then write down the vacuum solution as well as physically explain how the solution differs from one with the static condition [3].

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6 Black hole thermodynamics

Question 6.

For the second law of black hole mechanics, explain in detail as following

- 6.1)** [3 points.] Express/write down the statement for the second law of black hole mechanics.
- 6.2)** [3 points.] Explain how to obtain such the statement by using the Penrose process
- 6.3)** [4 points.] Explain how to obtain such the statement by using the focusing theorem

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