



Qualifying Examination



Academic Year 2024 (2567) : PART 1 : ROUND 3

13 MAY 2025

Name:.....

Student ID:.....

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 8 questions .
2. Write down detailed and clear solution to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	
7	
8	

1 Vector Calculus

Question 1.

Consider the Poisson equation with the spherical symmetry:

$$\nabla^2 \Phi(r) = S(r), \quad (1)$$

where Φ and S are the potential and source terms which depend only on the radial coordinate r .

1.1) [3 points.] Find the solution for Eq. (1) using direct integration.

1.2) [3 points.] Find the solution for Eq. (1) using Gauss theorem

1.3) [2 points.] Simplify the solution obtained in the previous problems by supposing that S does not depend on r

1.4) [2 points.] Derive the Coulomb law, i.e., force is proportional to $1/r^2$, from the previous result

Name:.....

Name:.....

Name:.....

2 Linear Algebra

Question 2.

2.1) [5 points.]

Calculate the eigenvalues of matrix P satisfying the condition

$$P^2 = P. \quad (2)$$

2.2) [5 points.]

Let A be the 2×2 matrix. Show that one of the eigenvalues of this matrix vanishes if the determinant of this matrix vanishes

Name:.....

Name:.....

3 Group Theory/Lie Algebra

Question 3.

Let G be a finite group of order n such that all its non-trivial elements have order 2.

3.1) Show that G is abelian. [2 points.]

3.2) Let H be a subgroup of G , and let $g \in G$ but not in H . Show that $H \cup gH$ is a subgroup of G . [2 points.]

3.3) Show that the subgroup $H \cup gH$ has order twice the order of H . [3 points.]

3.4) Deduce from the previous steps that the order of G is a power of 2. [3 points.]

Name:.....

Name:.....

4 Group Theory/Lie Algebra

Question 4. [10 points.] Let $p, q \in \mathbb{N}$ and $n = p + q$. We define the (indefinite) symmetric bilinear form $\langle \cdot, \cdot \rangle_{p,q}$ of signature (p, q) on $\mathbb{R}^n$ to be

$$\langle v, w \rangle_{p,q} := v_1 w_1 + \cdots + v_p w_p - v_{p+1} w_{p+1} - \cdots - v_{p+q} w_{p+q},$$

for all $v = (v_1, \dots, v_n)$, $w = (w_1, \dots, w_n) \in \mathbb{R}^n$. As the orthogonal group $O(n)$ is defined to be the group of matrices that preserve the standard Euclidean inner product we may now define $O(p, q)$ to be the group of matrices that preserve the above bilinear form:

$$O(p, q) := \{A \in GL(n, \mathbb{R}) : \langle Av, Aw \rangle_{p,q} = \langle v, w \rangle_{p,q} \forall v, w \in \mathbb{R}^n\}$$

Show that $O(p, q)$ is a Lie group using the inverse function theorem/constant rank theorem. What is its dimension?

Name:.....

Name:.....

5 Function of Complex Variables

Question 5.

5.1 [6 points.] Each of the following functions f has an isolated singularity at $z = 0$. Determine its nature; if it is a removable singularity define $f(0)$ such that f is analytic at $z = 0$; if it is a pole find the singular part; if it is an essential singularity determine $f(\{z : 0 < |z| < \delta\})$ for arbitrarily small values of δ .

- 5.1.1 [2 points.] $f(z) = \frac{\sin z}{z}$
- 5.1.2 [2 points.] $f(z) = \frac{\cos z}{z}$
- 5.1.3 [2 points.] $f(z) = \frac{\ln(1+z)}{z^2}$

5.2 [4 points.] With the help of contour integral and residue theorem, evaluate

$$\int_0^\infty \frac{\cos x - 1}{x^2} dx$$

Name:.....

Name:.....

6 Integral Transforms

Question 6.

Let us consider the following differential equation which, in the WKB method, allows us to find one of the WKB connection formulas that relate an exponential solution on one side of a turning point to an oscillatory solution on the other side:

$$\frac{d^2}{dx^2}\psi(x) + x\psi(x) = 0.$$

Here, we want to write the solution of this equation in integral form via the Fourier transform method. We recall that the solution $\psi(x)$ can be written as follows:

$$\psi(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dk \phi(k) e^{+ikx},$$

where $\phi(k)$ is its Fourier transform, namely,

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \psi(x) e^{-ikx}.$$

6.1) Multiply the differential equation by e^{-ikx} and integrate with dx . Find the differential equation that $\phi(k)$ must satisfy (assuming that $\psi(x)$ and $d\psi(x)/dx$ go to zero sufficiently rapidly at infinity).

[5 points.]

6.2) Solve the differential equation you found in the previous question and substitute its general solution into the integral expression for $\psi(x)$ given above.

[5 points.]

Final note: To obtain the WKB connection formula, one only needs to calculate separately the asymptotic forms for $x \rightarrow -\infty$ and $x \rightarrow +\infty$ of $\psi(x)$. The integral you have obtained (provided you have done your work well) is usually called Airy integral, and it is not a trivial integral.

Name:.....

Name:.....

7 Integral Transforms

Question 7.

If $\psi(x)$ is a (real or complex) function of the variable x , its Fourier transform $\phi(k)$, if it exists, is defined by

$$\phi(k) \equiv \mathcal{F}[\psi(x)] = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} dx \psi(x) e^{-ikx}.$$

7.1) Demonstrate the following property:

$$\mathcal{F}[\psi(cx)] = \frac{1}{|c|} \phi\left(\frac{k}{c}\right),$$

where c is an arbitrary real constant, i.e., $c \in \mathbb{R}$.

[3 points.]

7.2) From the above result, show that if the function $\psi(x)$ has a definite parity, its Fourier transform has the same parity. In other words, if $\psi(x)$ is an even (odd) function, then $\phi(k)$ is also an even (odd) function.

[3 points.]

7.3) If the function $\psi(x)$ is a real function, is its Fourier transform also real? Justify your answer!

[2 points.]

7.4) If the function $\psi(x)$ is a pure imaginary function, is its Fourier transform also pure imaginary? Justify your answer!

[2 points.]

Name:.....

Name:.....

8 Ordinary Differential Equations

Question 8.

Let us consider the following eigenvalue problem:

$$\frac{d^2}{dx^2}\psi(x) + E\psi(x) = 0, \quad \psi(0) = \psi(1) \quad \text{and} \quad \left(\frac{d\psi}{dx}\right)(0) = \left(\frac{d\psi}{dx}\right)(1),$$

where $0 \leq x \leq 1$ and E is a real parameter (i.e., $-\infty < E < +\infty$). We are interested in the nontrivial solutions of this problem, which appear only when the parameter E takes special values called eigenvalues.

8.1) Find the eigenvalues of the problem. Hint: Because $E \in \mathbb{R}$, it is appropriate to analyze the problem while separately considering the cases $E = 0$, $E < 0$ and $E > 0$.

[5 points.]

8.2) Write explicitly the nontrivial solutions of the problem, i.e., the eigenfunctions $\psi = \psi(x)$ corresponding to the eigenvalues E obtained in the previous question.

[5 points.]

Name:.....

Name:.....
