



Qualifying Examination



Academic Year 2024 (2567) : PART 2 : ROUND 3
15 MAY 2025

Name:.....

Student ID:.....

Part 2: Subjects Relating to student's Research Field : Quantum Information Science

Instructions and cautions

1. Choose 3 out of the following 6 questions relating student's field.
2. Write down detailed and clear solution to problems of your choice.
3. Complete answer are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	

1 Mathematical structure of quantum information

Question 1.

1.1) **Find** the (real) dimension of the special unitary group $SU(d)$, the group of d -dimensional unitary matrices with determinant 1. [2 points]

1.2) Consider density operators on a d -dimensional Hilbert space $\mathcal{H}_A$. **How many** real parameters are required to describe the set of density operators of rank $r \leq d$? (Of course, by setting the parameters to certain values, the parametrized density operator may have rank less than r . But the set of such values has measure zero in the set of rank- r density operators, and hence can be ignored for parameter counting.) Note that your answer should correctly reduce to those in the limiting cases: $2d - 2$ for pure states ($r = 1$), and $d^2 - 1$ for full-rank states ($r = d$). [5 points]

1.3) The minimal dimension of an ancillary Hilbert space required to purify a rank- r density operator is r . So let us consider the bipartite Hilbert space $\mathcal{H}_{AB} = \mathcal{H}_A \otimes \mathcal{H}_B$ with $\dim \mathcal{H}_B = r$ to be the space of purifications of rank- r density operators. Pure states in $\mathcal{H}_{AB}$ are described by $2dr - 2$ real parameters. **Verify** that subtracting $2dr - 2$ by your answer in 1.2) gives the dimension of $SU(r)$. **Explain** this observation from the point of view of the HJW theorem. [3 points]

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2 Quantum computation and algorithms

Question 2.

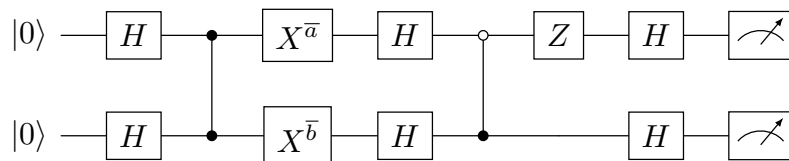
2.1) Consider the two-qubit ($N = 4$) version of Grover search when there is only one solution ($M = 1$). In this case, a single Grover iteration suffices to identify with certainty which one of the four Boolean functions,

$$f(a, b) = \begin{cases} 1, & ab \text{ is the solution,} \\ 0, & \text{otherwise,} \end{cases}$$

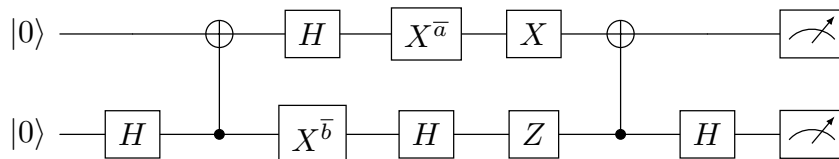
is implemented by the oracle. **Write** circuits that query the four Boolean functions using a Toffoli gate surrounded by bit flips as needed. [2 points]

2.2) **Write** the complete three-qubit circuit for Grover search in this setting. [2 points]

2.3) The third qubit remains unchanged throughout the computation and only serves to enable phase kickback. Employing any circuit identities at your disposal, **reduce** your circuit to an equivalent two-qubit circuit, and ultimately **convert** the circuit to that for superdense coding. (**Hint:** First reduce the three-qubit circuit to the following circuit:



which in turn is equivalent to the following circuit:



Then identify the Bell state $(|00\rangle + |11\rangle)/\sqrt{2}$ and the Bell measurement at the beginning and the end of the circuit. [6 points]

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3 Quantum information theory, distinguishability measures

Question 3.

3.1) Suppose Alice wants to approximately clone an unknown qubit state $|\psi\rangle$. Her strategy is to perform a projective measurement onto an orthonormal basis $\{|e_0\rangle, |e_1\rangle\}$ on her qubit state, and send the classical description of her post-measurement state to Bob. Bob then simply reconstructs the post-measurement state. **Show** that the average fidelity over the unknown initial state is

$$\overline{F} \equiv \int d\psi p(\psi) \sum_{j=0}^1 p(j|\psi) |\langle e_j|\psi\rangle|^2 = \frac{2}{3},$$

where $p(\psi)$ is the normalized ($\int d\psi p(\psi) = 1$) uniform distribution over all qubit pure states, and $p(j|\psi)$ is the probability of Alice obtaining the outcome j . (You may use the identity $\cos^2(\theta/2) = (1 + \cos \theta)/2$.) [3 points]

3.2) Now consider an approximate cloning machine acting on three qubits according to

$$\begin{aligned} |000\rangle_{ABC} &\mapsto \sqrt{\frac{2}{3}} |000\rangle_{ABC} + \sqrt{\frac{1}{3}} |\Psi^+\rangle_{AB} |1\rangle_C, \\ |100\rangle_{ABC} &\mapsto \sqrt{\frac{2}{3}} |111\rangle_{ABC} + \sqrt{\frac{1}{3}} |\Psi^+\rangle_{AB} |0\rangle_C, \end{aligned}$$

where $|\Psi^+\rangle = (|01\rangle + |10\rangle)/\sqrt{2}$. Is such a machine physically realizable in principle? Why or why not? [1 point]

3.3) The action given in 3.2) performs approximate cloning because the resulting reduced states ρ'_A and ρ'_B are identical, but imperfect, copies of the input state $|\psi\rangle_A$. The mapping from the initial state $|\psi\rangle_A \langle\psi|$ to ρ'_A defines a superoperator $\mathcal{A}$. **Find** an operator-sum (Kraus) representation of $\mathcal{A}$. [3 points]

3.4) For $|\psi\rangle_A = \alpha|0\rangle + \beta|1\rangle$, **compute** the fidelity $\langle\psi|\mathcal{A}(|\psi\rangle\langle\psi|)|\psi\rangle$ between the input state and the output state. Is the fidelity greater than $2/3$? [3 points]

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4 Many-body physics in second quantized formalism

Question 4.

Let a_1, a_2 be two fermionic annihilation operators.

4.1) Show that the Bogoliubov transformation

$$\begin{aligned}c_1 &= ua_1 + va_2^\dagger, \\c_2^\dagger &= -va_1 + ua_2^\dagger,\end{aligned}$$

where u and v are real numbers, preserves the canonical anticommutation relations if $u^2 + v^2 = 1$ [4 points]

4.2) Use the result in 4.1) to show that the Hamiltonian

$$H = \epsilon \left(a_1^\dagger a_1 - a_2 a_2^\dagger \right) + \Delta \left(a_1^\dagger a_2^\dagger + a_2 a_1 \right)$$

can be diagonalized in the form

$$H = \sqrt{\epsilon^2 + \Delta^2} \left(c_1^\dagger c_1 + c_2^\dagger c_2 - 1 \right).$$

[2 points]

4.3) What is the ground-state energy of this Hamiltonian? [2 points]

4.4) Write the ground-state wavefunction in terms of the original operators $c_1^\dagger$ and $c_2^\dagger$ and their corresponding vacuum $|0\rangle$ ($c_{1,2}|0\rangle = 0$). [2 points]

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5 Generalized measurement (POVM) and Quantum Channels

Question 5.

Consider a quantum system known to be in one of two pure states, $|\psi_{-1}\rangle$ or $|\psi_1\rangle$, with prior probabilities $1/2$ each. We label the hypotheses by H_{-1} : “the system is in state $|\psi_{-1}\rangle$ ” and H_1 : “the system is in state $|\psi_1\rangle$.” Our goal is to measure the system so as *never* to make a misidentification: if the measurement outcome declares “the state was $|\psi_{-1}\rangle$,” it must indeed have been $|\psi_{-1}\rangle$; and similarly, if the outcome declares “the state was $|\psi_1\rangle$,” it must have been $|\psi_1\rangle$. Because these two states need not be orthogonal, we allow an additional, third outcome-, which we will call *error* or *inconclusive*, that admitted we cannot be sure which state we had.

Specifically, suppose we work in a two-dimensional Hilbert space $\mathcal{H}$ spanned by an orthonormal basis $\{|x\rangle, |y\rangle\}$. Let $0 < \theta < \pi/4$ and define

$$|\psi_{-1}\rangle = \cos(\theta) |x\rangle - \sin(\theta) |y\rangle, \quad |\psi_1\rangle = \cos(\theta) |x\rangle + \sin(\theta) |y\rangle. \quad (1)$$

We also define the kets

$$|\xi_{-1}\rangle = -\sin(\theta) |x\rangle + \cos(\theta) |y\rangle, \quad |\xi_1\rangle = -\sin(\theta) |x\rangle - \cos(\theta) |y\rangle. \quad (2)$$

Consider the three operators

$$\hat{\Pi}_{-1} = a |\xi_{-1}\rangle \langle \xi_{-1}|, \quad \hat{\Pi}_1 = a |\xi_1\rangle \langle \xi_1|, \quad \hat{\Pi}_e = b |x\rangle \langle x|, \quad (3)$$

where a and b are positive real constants to be determined.

5.1) Find a and b so that $\{\hat{\Pi}_{-1}, \hat{\Pi}_1, \hat{\Pi}_e\}$ is a valid positive-operator-valued measure (POVM). That is, show how to choose a and b so that each $\hat{\Pi}_j$ is positive semidefinite and

$$\hat{\Pi}_{-1} + \hat{\Pi}_1 + \hat{\Pi}_e = \hat{I}_2, \quad (4)$$

where $\hat{I}_2$ is the 2×2 identity operator on $\mathcal{H}$.

[4 points]

5.2) When the measurement described by $\{\hat{\Pi}_{-1}, \hat{\Pi}_1, \hat{\Pi}_e\}$ is performed on a state $|\psi\rangle \in \mathcal{H}$, the outcome is “−1” with probability $\langle \psi | \hat{\Pi}_{-1} | \psi \rangle$, “1” with probability $\langle \psi | \hat{\Pi}_1 | \psi \rangle$, or “error” with probability $\langle \psi | \hat{\Pi}_e | \psi \rangle$. Declare “the state was $|\psi_{-1}\rangle$ ” if the outcome is −1, “the state was $|\psi_1\rangle$ ”

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if the outcome is 1, and “error” otherwise. Show that if the measurement outcome is -1 , the system *could not* have been in $|\psi_1\rangle$, and if the outcome is 1, the system *could not* have been in $|\psi_{-1}\rangle$. Hence no misidentification ever occurs when the measurement reports one of the two candidate states. [3 points]

5.3) Assuming the two hypotheses H_{-1} and H_1 both occur with prior probability $1/2$, find the probability that the measurement outcome is “error.” [3 points]

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6 Quantum Approximate Optimization Algorithm (QAOA)

Question 6.

The QAOA circuit at depth p consists of alternative layers of unitary operators:

$$|\gamma, \beta\rangle = \prod_{k=1}^p e^{-i\beta_k B} e^{-i\gamma_k C} |+\rangle^{\otimes n},$$

where $B = \sum_{l=1}^n X_l$ is the *mixing Hamiltonian*, and C is the *cost Hamiltonian* which depends on the optimization problem considered.

6.1) Discuss how the choices of γ_k and β_k effect exploration and exploitation in QAOA.

[3 points]

Now consider QAOA for the MaxCut problem on 2-node graphs. In this case, QAOA can find the exact solution with $p = 1$, and the optimal parameters can be solved analytically.

6.2 Write the cost Hamiltonian and give the form of a quantum state that optimizes the cost function.

[2 points]

6.3) It turns out that, up to the global phase, the final state $|\gamma, \beta\rangle$ of the depth-1 QAOA can be written as

$$\frac{1}{2} \begin{bmatrix} \cos q_+ - i \sin q_- \\ \cos q_- - i \sin q_+ \\ \cos q_- - i \sin q_+ \\ \cos q_+ - i \sin q_- \end{bmatrix},$$

where

$$q_{\pm} = 2\beta \pm \frac{\gamma}{2}.$$

Based on your solution to 6.2), give values γ^*, β^* that provide an optimal state. [2 points]

6.4) Verify directly (without using the form of the final state in 6.3) that the expectation value $\langle \gamma^*, \beta^* | C | \gamma^*, \beta^* \rangle$ attains the optimum. (You may use the identity $e^{i\beta X} Z e^{-i\beta X} = Z \cos(2\beta) + Y \sin(2\beta)$.) [3 points]

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