



Qualifying Examination



Academic Year 2025 (2568) : PART 1 : ROUND 3

13 MAY 2026

Name:.....

Student ID:.....

Part 1: Mathematical Methods of Physics - Compulsory

Instructions and cautions

1. Choose 4 out of the following 8 questions .
2. Write down detailed and clear solution to problems of your choice.
3. Complete answers are preferred to fragments.
4. Write on one side of the paper only and begin each answer on a space provided.
5. Write legibly, otherwise you place yourself at a grave disadvantage.

Question No.	Checked Question to be marked
1	
2	
3	
4	
5	
6	
7	
8	

1 Vector Calculus

Question 1.

Consider three-dimensional flat spacetime with Cartesian coordinates (x^1, x^2, x^3) . Let

$$\vec{E} = \frac{e^{-\mu r}}{r^3} \vec{r},$$

where $\vec{r} = x^1 \hat{e}_1 + x^2 \hat{e}_2 + x^3 \hat{e}_3$, and $r \equiv \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}$.

1.1) [1 point.] Consider an area element of $S_R^2 = \{(x^1, x^2, x^3) \mid (x^1)^2 + (x^2)^2 + (x^3)^2 = R^2\}$, a spherical surface of radius R . Write down the unit vector $\hat{n}$ perpendicular to this area element and pointing radially outward.

1.2) [2 points.] Evaluate

$$\int_{S_R^2} dS \vec{E} \cdot \hat{n},$$

where dS is the area element of S_R^2 .

1.3) [2 points.] Compute $\vec{\nabla} \cdot \vec{E}$ for $r > 0$.

1.4) [3 points.] Evaluate

$$\int_{\mathcal{V}_{\epsilon, R}} dV \vec{\nabla} \cdot \vec{E}, \quad 0 < \epsilon < R,$$

where dV is a volume element of $\mathcal{V}_{\epsilon, R} = \{(x^1, x^2, x^3) \mid \epsilon^2 \leq (x^1)^2 + (x^2)^2 + (x^3)^2 \leq R^2\}$.

1.5) [2 points.] With the help of divergence theorem, evaluate

$$\int_{\mathcal{V}_{0, \epsilon}} dV \vec{\nabla} \cdot \vec{E},$$

where $\mathcal{V}_{0, \epsilon} = \{(x^1, x^2, x^3) \mid 0 \leq (x^1)^2 + (x^2)^2 + (x^3)^2 \leq \epsilon^2\}$

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2 Linear Algebra

Question 2.

2.1) [3 points.] Let M be a 2×2 matrix. Show that its eigenvalues λ satisfies the relation

$$\lambda^2 - \operatorname{tr}(M)\lambda + \det(M) = 0.$$

2.2) [5 points.] Suppose M is a real 2×2 matrix that satisfies $M^2 = -I$, where I is the identity matrix. Determine the eigenvalues of M . Consequently, find $\operatorname{tr}(M)$ and $\det(M)$.

2.3) [2 points.] Give an example of a real 2×2 matrix M that satisfies $M^2 = -I$. Can there exist a real 3×3 matrix N satisfying $N^2 = -I$? If yes, give an example. If no, show why it cannot exist.

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3 Group Theory/Lie Algebra

Question 3.

Consider the following set of 2×2 real matrices:

$$S = \left\{ A_\theta = \begin{pmatrix} \cos \theta & \sin \theta \\ \sin \theta & -\cos \theta \end{pmatrix} \mid \theta \in \mathbb{R} \right\}$$

3.1) [2 points.] Show that S does **not** form a group under matrix multiplication. Provide at least two violated group axioms to justify your answer.

3.2) [4 points.] Denote by R_ϕ the product of two matrices from this set, $A_{\theta_1} A_{\theta_2}$, where the angle ϕ is related to θ_1 and θ_2 . Let S' be the set of such R_ϕ for all $\phi \in \mathbb{R}$. Prove that the extended set $G = S \cup S'$ now satisfies the closure and identity axioms under matrix multiplication.

3.3) [1 point.] What is the standard mathematical notation for the group G ?

3.4) [3 points.] Characterize the Lie algebra $\mathfrak{g}$ of G . Can we obtain every element of G by exponentiating the elements of the Lie algebra from the identity element (i.e., $A = e^C$ for all $C \in \mathfrak{g}$)? Give a reason why or why not.

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4 Group Theory/Lie Algebra

Question 4.

4.1) [3 points.] For elements A and B in a Lie algebra, the adjoint action is defined as $\text{ad}_A B = [A, B]$. By considering the Taylor expansion of the function $f(\lambda) = e^{\lambda A} B e^{-\lambda A}$ around $\lambda = 0$, show that

$$e^A B e^{-A} = e^{\text{ad}_A} B.$$

4.2) [7 points.] If A and B both commute with their commutator (i.e., $[A, [A, B]] = [B, [A, B]] = 0$), determine the value of k in the following equation:

$$e^{A+B} = k e^A e^B.$$

Hint: Consider the function $f(\lambda) = e^{\lambda(A+B)} e^{-\lambda B} e^{-\lambda A}$. Establish the differential equation $\frac{df}{d\lambda} = -\lambda[A, B]f$, and integrate this equation.

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5 Function of Complex Variables

Question 5.

5.1) [4 points.] Let D be an open subset of $\mathbb{C}$ and $f : D \rightarrow \mathbb{C}, u : D \rightarrow \mathbb{R}, v : D \rightarrow \mathbb{R}$. The function f is analytic in D if it can be expressed as $f(x, y) = u(x, y) + iv(x, y)$, where u and v have continuous partial derivatives $\partial u/\partial x, \partial u/\partial y, \partial v/\partial x, \partial v/\partial y$ which satisfy Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}.$$

Consider the case where $D = \mathbb{C} \setminus \{0\}$, $u(x, y) = x/(x^2 + y^2)$. Find $v(x, y)$ which makes $f(x, y)$ analytic in D , and that $v(1, 0) = i$.

5.2) [6 points.] With the help of contour integral and residue theorem, evaluate

$$\int_{-\infty}^{\infty} \frac{\cos(kx)}{x^2 + 1} dx,$$

where $k \geq 0$.

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6 Integral Transforms

Question 6.

A Fourier transform of a function $f(x)$ is given by

$$\mathcal{F}(f(x)) = F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx f(x) e^{ikx}$$

6.1) [2 points.] show that the Fourier transform of the n -derivative of $f(x)$ is

$$\mathcal{F}\left(\frac{d^n f(x)}{dx^n}\right) = (-ik)^n F(k).$$

6.2) [8 points.] Use the Fourier transform to solve the heat equation

$$\frac{\partial}{\partial t} u(x, t) = \alpha \frac{\partial^2}{\partial x^2} u(x, t).$$

Show that the solution $u(x, t)$ of the heat equation with initial data $u(x, 0) = \phi(x)$ is

$$u(x, t) = \frac{1}{\sqrt{4\pi\alpha t}} \int_{-\infty}^{\infty} e^{-\frac{(x-y)^2}{4\alpha t}} \phi(y) dy.$$

Hint: The following relations may be useful:

- Convolution:

$$\mathcal{F}^{-1}(F(k)G(k)) = \frac{1}{\sqrt{2\pi}} (f * g)(x),$$

where the convolution $(f * g)(x)$ is defined by

$$(f * g)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x-y)g(y) dy.$$

$F(k)$ and $G(k)$ are Fourier transforms of $f(x)$ and $g(x)$ respectively.

- Inverse Fourier transform of Gaussian function:

$$\mathcal{F}^{-1}\left(e^{-\alpha k^2 t}\right) = \frac{1}{\sqrt{4\alpha t}} e^{-\frac{x^2}{4\alpha t}}.$$

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7 Integral Transforms

Question 7.

A particle of mass m moves along the x -axis subject to a linear restoring force $-kx$ and a damping force proportional to its velocity $-b\frac{dx}{dt}$. The displacement $x(t)$ of the particle satisfies the second-order linear differential equation:

$$m\frac{d^2x}{dt^2} + b\frac{dx}{dt} + kx = F(t),$$

where $F(t)$ is an external driving force. Assume the particle has initial data $x(0) = 1$ and $x'(0) = 0$.

7.1) [3 points]

The Laplace transform of a function $f(t)$ is defined by

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t) dt.$$

Using this definition, find the Laplace transforms of the following functions:

$$t, \quad e^{at}, \quad \sin(\omega t),$$

where a is a constant and $\omega > 0$.

7.2) [4 points]

Let $F(s) = \mathcal{L}\{f(t)\}$. Show that the Laplace transform of the first and second derivatives, $f'(t)$ and $f''(t)$ are given by

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0).$$

and

$$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0).$$

7.3) [3 points]

Consider a system with $m = 1$, $b = 3$, and $k = 2$. If there is no external force $F(t)$ applied to the system. Apply the Laplace transform to the differential equation and solve for the displacement $x(t)$.

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8 Ordinary Differential Equations

Question 8.

Consider a second-order linear homogeneous differential equation in standard form:

$$y'' + p(x)y' + q(x)y = 0.$$

If one non-zero solution $y_1(x)$ is known, a second linearly independent solution $y_2(x)$ can be found using the properties of the Wronskian, $W(x) = W(y_1, y_2)$.

8.1) [2 points]

Show that the Wronskian

$$W(x) = W(y_1, y_2) = y_1 y_2' - y_1' y_2$$

satisfies a first-order differential equation

$$W' + p(x)W = 0.$$

Then show that:

$$W(x) = C \exp \left(- \int p(x) dx \right),$$

where C is a constant.

8.2) [3 points]

Show that the second solution $y_2(x)$ can be expressed as:

$$y_2(x) = y_1(x) \int \frac{W(x)}{[y_1(x)]^2} dx.$$

8.3) [5 points]

Consider the differential equation for $x > 0$:

$$x^2 y'' - x(x+2)y' + (x+2)y = 0.$$

Verify that $y_1(x) = x$ is a solution to the equation. Then use the Wronskian method to find the general solution $y_2(x)$.

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